Achieving the Lower Bound of Fading MIMO Relay Channels with Covariance Feedback at the Transmitters

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Abstract—In this paper, we consider a full-duplex mode MIMO relay channel for decode and forward strategy where the receivers have perfect channel state information (CSI) and the transmitters have only covariance feedback at the transmitters. We derive a lower bound to the ergodic capacity for this scenario and propose an iterative algorithm that finds lower bound achieving transmit covariance matrices of the source and relay nodes. The solution of the optimization problem in the lower bound expression is given for three cases depending on channel covariance matrices. For one of these cases, it seems that an efficient and fast algorithm achieving the lower bound is not possible. However, under a certain practical channel assumption, we propose a power allocation algorithm that gives a solution much faster than classical convex optimization methods. Moreover, we show that this fast algorithm results in a data rate which is very close to the lower bound to the capacity.

Index Terms—MIMO relay channels, covariance feedback, decode and forward.

I. INTRODUCTION

Over the last decade, multi antenna systems became very popular since they increase the available spectral efficiency with the same amount of transmit power. However, this increase in data rate crucially depends on the amount of channel knowledge at the transmitters and receivers. In [1], a single-user MIMO system is considered when both the receiver and transmitter have perfect CSI and the channel is fixed. In this case, the optimum power allocation is to water-fill over the singular values of the deterministic channel matrix. In [2], a multi-user MIMO system is considered when all the transmitters and the receiver have perfect CSI and the channels are fixed. In this scenario, the optimum transmit directions and the power allocation policies are found using an iterative algorithm. However, the case that all the transmitters and the receiver have perfect CSI is not practical. In practice, the receiver feeds the information back to the transmitters and for a fast fading wireless channel, it is not possible to send perfect instantaneous feedback to the transmitters. Therefore, it is more realistic that the receivers have perfect CSI and the transmitters have only a statistical knowledge of the channel [3]–[5]. If the channel follows a Gaussian process, statistics of the channel are the mean and covariance information.

In addition to utilizing multiple antennas to increase the capacity of the system, adding a relay node and applying cooperative strategies can also increase capacity. In [6], a detailed description of the relay channel is given with different scenarios. Although the capacity is not known in general, upper and lower bounds can be derived. In decode and forward (DF) relay systems, the relay node demodulates and decodes the received signal from the source node, and retransmits it to the destination node. In [7], [8] the ergodic rate and outage probability are studied for single antenna relay channels with DF operation. The results show that optimum relay channel signaling outperforms multihop protocols. In [9], a max-min type of problem is introduced for fading relay channels. Bounds on channel capacity are derived for synchronized and asynchronous cases. In [10], single user MIMO relay channels are presented when both the receivers and transmitters have perfect CSI. In [11], a more realistic scenario is considered where only the receiver side knows the perfect CSI and transmitters do not know the channel. Moreover, it is found in [11] that the channel inputs of the source and relay nodes are independent when the channel is fading.

In this paper, we consider a MIMO relay channel in DF mode when the receivers have the perfect CSI and the transmitters have only the statistics of the channel. The capacity of the MIMO relay channel under such an assumption is not known in general, but lower and upper bounds to the capacity can be derived. For the model in this paper, we derive a lower bound in terms of a max-min problem and solve this problem using similar techniques in [9]. In this technique, we describe lower bound in three cases. In the first case, lower bound on the capacity is equal to the capacity of the link from source to relay. In the second case, lower bound on the capacity is equal to the multi access channel capacity from source and relay to the destination. The optimization problems in the first and second cases can be solved by developing fast and efficient algorithms (similar to [5]) in order to solve for the transmit covariance matrices. In the last case, the lower bound to the capacity depends on both multi access channel and source to relay channel. The optimization problem in the third case can be solved using classical convex optimization techniques [12], but developing a fast and efficient algorithm does not seem to be possible. However, by exploiting the nature of the relay channel and assuming that the source to destination channel is weaker than the source to relay channel, we are able to propose a fast and efficient algorithm that results in a rate which is very close to the lower bound. The proposed algorithm converges much faster than the classical convex optimization methods.

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The relay node is assumed to operate in full-duplex mode. As \( y = H_{sd} x_s + H_{rd} x_r + n_d \) to the destination node and \( x_r \) is an \( M_r \times 1 \) transmitted signal from the relay node to the destination node. The covariance matrices of the transmitted signals are \( Q_s = E[x_s x_s^H] \) and \( Q_r = E[x_r x_r^H] \). The received signal at the destination node, \( y \), is \( N_d \times 1 \). The received signal at the relay node, \( r \), is \( N_r \times 1 \). The relay node is assumed to operate in full-duplex mode. As shown in Figure 1, \( H_{sr}, H_{sd} \) and \( H_{rd} \) are \( N_r \times M_s, N_d \times M_s \) and \( N_d \times M_r \) dimensional channel matrices. Noise vectors at the relay, \( n_r \), and at the destination, \( n_d \) are zero-mean, identity covariance complex Gaussian random vectors.

**II. SYSTEM MODEL**

We consider a MIMO relay channel when the receivers have perfect CSI and the transmitters only have the transmit covariance information. The channel between a transmitter and a receiver is represented by a random matrix \( H_{xy} \) whose dimensions are the number of receive antennas times the number of transmitter antennas. In the case that the receiver has perfect CSI and the transmitter has only transmit covariance information, there is a correlation between the signals transmitted by or received at different antenna elements. The channel model is defined as \( y = H_{xy} x + n \) where subscript \( xy \) refers to either \( sr \) (source to relay), \( sd \) (source to destination), or \( rd \) (relay to destination); \( Z_{xy} \) is an identity covariance random channel matrix, \( \Sigma_{xy} \) is the correlation matrix between the signals transmitted from the antennas on the transmitter.

In the cut set upper bound, the links from source to relay and destination are not exact broadcast channel (BC) since the same signal is transmitted both relay and destination nodes. On the other hand, this channel is like BC since the signal is transmitted from one transmitter to two receivers. The links from relay and source to destination node are exact Multi Access Channel (MAC) as shown in Figure 1. The received signals at the relay and destination nodes for general MIMO relay channels are defined as \( r = H_{sr} x_s + n_r \) and \( y = H_{sd} x_s + H_{rd} x_r + n_d \)

where \( x_s \) is an \( M_s \times 1 \) transmitted signal from the source node to the destination node and \( x_r \) is an \( M_r \times 1 \) transmitted signal from the relay node to the destination node. The covariance matrices of the transmitted signals are \( Q_s = E[x_s x_s^H] \) and \( Q_r = E[x_r x_r^H] \). The received signal at the destination node, \( y \), is \( N_d \times 1 \). The received signal at the relay node, \( r \), is \( N_r \times 1 \). The relay node is assumed to operate in full-duplex mode. As shown in Figure 1, \( H_{sr}, H_{sd} \) and \( H_{rd} \) are \( N_r \times M_s, N_d \times M_s \) and \( N_d \times M_r \) dimensional channel matrices. Noise vectors at the relay, \( n_r \), and at the destination, \( n_d \) are zero-mean, identity covariance complex Gaussian random vectors.

**III. LOWER BOUND ON THE CAPACITY**

When the receivers have perfect CSI and the transmitters only have transmit covariance matrix at the transmitters, the channel capacity is not known in general, but lower and upper bounds on the capacity can be found. In this paper, we find a lower bound to the MIMO relay channel capacity for the decode and forward strategy in terms of the capacity of the link from the source to relay, capacity of MAC from source and relay to the destination and capacity of the link from the relay to destination. This lower bound involves a max-min type problem [9]. This problem is solved by choosing the transmit covariance matrices of the source and the relay for each of the three cases.

Since our results will depend on single-user MIMO and MIMO-MAC mutual information expressions, here we state them for the sake of completeness. Single user MIMO channel capacity where the receiver has perfect CSI and the transmitter has only statistical knowledge of channel is known as [5]

\[
C = \max_{w(Q)} E \left[ \log |I + HQH^H| \right] \tag{4}
\]

where \( E[\cdot] \) is the expectation operator with respect to the channel matrices \( |\cdot| \) is the determinant operator, \( \text{tr}(\cdot) \) is the trace of a matrix, and \( P \) is power constraint of the channel. Using eigenvalue decomposition, the channel covariance matrix and the transmit covariance matrix can be written, as \( \Sigma = U_s \Lambda_s U_s^H \) and \( Q = U_Q \Lambda_Q U_Q^H \) respectively where \( U_Q \) and \( U_s \) are unitary matrices, and \( \Lambda_s \) and \( \Lambda_Q \) are diagonal matrices that include ordered eigenvalues. Eigenvectors of transmit covariance matrix must be equal to the eigenvectors of the channel covariance matrix, \( U_Q = U_s \) [4]. The eigenvalues of the covariance matrix are optimized using the algorithm in [5].

The lower bound also depends on the two-user MAC channel, the capacity of which is defined as [5]

\[
C_{mac} = \max_{w(Q_s) \leq P_s, w(Q_r) \leq P_r} E \left[ \log |I + H_{sd} Q_s H_{sd}^H + H_{rd} Q_r H_{rd}^H| \right] \tag{5}
\]

where the receivers have perfect CSI and the transmitters have only transmit covariance information. The power constraint is \( P_s \) and \( P_r \) for the relay node. The eigenvectors of the transmit covariance matrices of each user depend only on the eigenvectors of their own channel covariance matrices, i.e., \( U_{Q_s} = U_{\Sigma_{sd}} \) and \( U_{Q_r} = U_{\Sigma_{rd}} \) [4]. The eigenvalues of the covariance matrices of the source and relay are found using the iterative algorithm in [5].

On the other hand, there are some results made on MIMO relay channels, when the receivers have perfect CSI and the transmitters have no CSI. Lower bound of the ergodic capacity in this case is found in [11]. We also state this result by assuming that the mutual information of source to destination
link is less than the mutual information of source to relay link.

\[
C \geq C_{lower} = \max_{\tr(Q_s) \leq P_s} \left( \min(C_{mac}, C_{sr}) \right) \tag{6}
\]

\[
C_{mac} = E \left[ \log \left| I + H_{sd}Q_sH_{sd}^H + H_{rd}Q_rH_{rd}^H \right| \right] \tag{7}
\]

\[
C_{sr} = E \left[ \log \left| I + H_{sr}Q_sH_{sr}^H \right| \right] \tag{8}
\]

Since the transmitters have no CSI, the lower bound is maximized by choosing \( x_s \) and \( x_r \) to be independent circular-symmetric vectors with \( Q_s = I \) and \( Q_r = I \).

In this study, when the receivers know the channel perfectly and the transmitters know the covariance information of the channel, we derive a lower bound on MIMO relay capacity. Theorem 1 gives this lower bound in terms of the capacity of the link from the source to relay, capacity of the MAC from source and relay to destination and the transmit covariance matrices at the source and relay.

**Theorem 1:** When there is only transmit covariance information at the transmitters and perfect CSI at the receivers, lower bound on ergodic capacity of a MIMO relay channel for DF strategy is given as

\[
C \geq C_{lower} = \max_{\tr(Q_s) \leq P_s} \min(I_{mac}, I_{sr}) \tag{9}
\]

with

\[
I_{mac} = E \left[ \log \left| I + H_{sd}Q_sH_{sd}^H + H_{rd}Q_rH_{rd}^H \right| \right] \tag{10}
\]

\[
I_{sr} = E \left[ \log \left| I + H_{sr}Q_sH_{sr}^H \right| \right] \tag{11}
\]

where \( Q_s = E[x_s x_s^H] \) and \( Q_r = E[x_r x_r^H] \) are transmit covariance matrices with \( \tr(Q_s) \leq P_s \) and \( \tr(Q_r) \leq P_r \).

**Proof:** Using block Markov coding technique, the achievable rate is written below [6]

\[
R = \max_{p(x_s, x_r)} \min(I(x_s; r | x_r), I(x_s, x_r; y)) \tag{12}
\]

and mutual information expressions in (12) can be written as

\[
I(x_s; r | x_r) = E[I(x_s; r | x_r, H_{sr})] \tag{13}
\]

\[
I(x_s, x_r; y) = E[I(x_s, x_r; y | H_{sd}, H_{rd})] \tag{14}
\]

where \( x_s \) and \( x_r \) are circularly-symmetric complex Gaussian random vectors. The expectation in (13) is calculated in [11] as

\[
I(x_s; r | x_r, H_{sr}) \leq \log \left| I + H_{sr}(Q_s - Q_sQ_r^{-1}Q_rQ_sH_{sr}) \right| \tag{15}
\]

\[
= \log \left| I + H_{sr}Q_sH_{sr}^H \right| \tag{16}
\]

The cross-correlation matrices \( Q_{sr} = E[x_s x_r^H] \) and \( Q_{rs} = E[x_r x_s^H] \) are zero to prevent the decrease of the mutual information values [10]. In other words the signals are independent. Using these definitions in (15), the single user capacity from the source to the relay node is achieved. This problem can be solved using previous techniques. Similarly, the expression in (14) is calculated as the following equation in [11]

\[
I(x_s, x_r; y | H) \leq \log \left| I + HQH \right| \tag{17}
\]

\[
= \log \left| I + H_{sd}Q_sH_{sd}^H + H_{rd}Q_rH_{rd}^H \right| \tag{18}
\]

where we define \( H = [H_{sr}, H_{rd}] \) and

\[
Q = [Q_s, Q_r] \tag{19}
\]

and we again used the fact that \( Q_{sr} = Q_{rs} = 0 \). Finally, we insert (16) and (18) into (12) and obtain (9)-(11). \( \square \)

It is important to note that the optimum \( Q_s \) maximizing \( I_{sr} \) and the optimum \( Q_r \) maximizing \( I_{mac} \) are different. If we maximize \( I_{mac} \), that choice of \( Q_s \) will result in a lower \( I_{sr} \). As a result, \( I_{sr} \) will come out of the minimization in (9), and the achievable rate will attain a lower value. As a solution to this, a max-min type of optimization is given in [9]. The following function \( R(\alpha) \) and \( Q \) is defined as

\[
R(\alpha, Q) = \alpha I_{mac}(Q) + (1 - \alpha)I_{sr}(Q), \quad 0 \leq \alpha \leq 1 \tag{20}
\]

where \( Q = [Q_s, Q_r] \). The maximization in (9) corresponds to the two end points of the line \( R(\alpha, Q) \) over all values of \( Q \).

\[
V(\alpha) = \max_Q R(\alpha, Q) \tag{21}
\]

which is equal to \( C_{lower} \) in (9). Depending on the value of \( \alpha^* \), we have three cases. In the first case \( \alpha^* = 0 \), \( R(0, Q) = I_{sr}(Q) \) and the condition \( I_{mac}(Q) \geq I_{sr}(Q) \) should be satisfied. Since the achievable rate is found by maximizing \( I_{sr}(Q) \) only, we find the optimum source transmit covariance matrix, \( Q_s \), as a solution to a single-user problem from source to relay. Then we find the optimum relay transmit covariance matrix, \( Q_r \), by maximizing \( I_{mac}(Q) \) with a fixed \( Q_s \) in order to satisfy \( I_{mac}(Q) \geq I_{sr}(Q) \).

In the second case, \( \alpha^* = 1 \), \( R(1, Q) = I_{mac}(Q) \) and the condition \( I_{mac}(Q) \leq I_{sr}(Q) \) should be satisfied. In this case, the achievable rate is found by maximizing \( I_{mac}(Q) \). Therefore, we find the optimum source transmit covariance matrix, \( Q_s \), and relay transmit covariance matrix \( Q_r \), as a solution to a MAC problem.

In the third case, \( 0 < \alpha^* < 1 \), \( R(\alpha^*, Q) = \alpha^* I_{mac}(Q) + (1 - \alpha^*)I_{sr}(Q) \) and the condition \( I_{mac}(Q) = I_{sr}(Q) \) should be satisfied. In this case, we find the optimum transmit covariance matrices of the source and relay as functions of \( \alpha^* \). This case is the most interesting case as the solution is not trivial.

It is important to note that the optimum source and relay covariance matrices may be different in all three cases. After finding the optimum transmit covariance matrices for a given \( \alpha \), for all \( 0 \leq \alpha \leq 1 \), we will find the optimum \( \alpha^* \) that minimizes \( V(\alpha) \). In the following, we present the derivation...
for all three cases in detail.

**Case 1:** When \( \alpha^* = 0 \), (20) is given by the mutual information of the link from the source to the relay as

\[
R(0, Q) = I_{sr}(Q)
\]

(23)

In addition, the following condition has to be satisfied [9]

\[
I_{mac}(Q) \geq I_{sr}(Q)
\]

(24)

When the receiver knows perfect CSI and the transmitter only knows covariance matrix at the transmitters, calculating the maximum mutual information from the source to the relay is a single user MIMO problem that is solved in [5].

However, the single-user solution only gives us the optimum \( Q_s \). In order to find \( Q_r \), only constraint we need to satisfy is the condition in (24). Therefore, we propose to find \( Q_r \) by maximizing \( I_{mac}(Q) \) with a fixed \( Q_s \). This is also a modified single-user problem, and single-user solution can easily be generalized to this case. The result is shown below. Here \( A_i = I_N + \sum_{i=1}^{M_r} \lambda_i^Q \Sigma_{sr}^{-1/2} z_{sri} z_{sri}^T \lambda_i^Q \Sigma_{sr}^{-1/2} \) is defined. The following algorithm can be derived similar to the single-user algorithm in [5].

\[
E_i(\lambda) \equiv E\left[ \sum_{i=1}^{M_r} \lambda_i^Q \Sigma_{sr}^{-1/2} z_{sri} z_{sri}^T \lambda_i^Q \Sigma_{sr}^{-1/2} \right]^2
\]

(25)

\[
\lambda_i^{Q^*}(n+1) = \frac{\lambda_i^{Q(n)} E_i(\lambda(n)) P_r}{\sum_{i=1}^{M_r} \lambda_i^{Q(n)} E_i(\lambda(n))}
\]

(26)

**Case 2:** When \( \alpha^* = 1 \), (20) is given by the mutual information of the link from the source to the relay as

\[
R(1, Q) = I_{mac}(Q)
\]

(27)

In addition, the following condition has to be satisfied [9]

\[
I_{mac}(Q) \leq I_{sr}(Q)
\]

(28)

When the receiver knows perfect CSI and the transmitters know the transmit covariance information, maximum mutual information for a MIMO-MAC system is solved in [5].

**Case 3:** When \( 0 < \alpha^* < 1 \), (20) transforms to following optimization problem.

\[
V(\alpha^*) = \max_{\text{tr}(Q) \leq P_r} \left( \alpha^* I_{mac}(Q) + (1 - \alpha^*) I_{sr}(Q) \right)
\]

(29)

In addition, the following condition has to be satisfied because of max-min rule [9]

\[
I_{mac}(Q) = I_{sr}(Q)
\]

(30)

In order to find the lower bound, one has to solve (29) for a given \( \alpha \), and search over \( 0 < \alpha < 1 \) to find \( V(\alpha^*) \). It seems that it is not possible to apply the methods of [5] directly to (29). On the other hand, it is always possible to solve (29) using classical convex optimization methods [12]. Disadvantage of classical convex optimization methods is that they are very slow, and therefore cannot be used in real-time communications in a fast fading wireless environment. However, under certain assumptions on the channel, it might be possible to propose fast and efficient algorithms. One such assumption is that source to destination link is weaker than the source to relay link. Therefore, source node chooses to transmit along the eigenvectors of the covariance of the source to relay channel, instead of the jointly optimal directions. Jointly optimal directions are possibly a combination of the eigenvectors of the covariances of the source to relay channel, and those of the source to destination channel. In vague terms, the source node chooses to transmit towards the relay.

Once the transmit directions of the source node is given, the transmit directions of the relay node can be found using the following theorem.

**Theorem 2:** Let us assume the channel covariance matrix from the relay to the destination, \( \Sigma_{rd} \), has the eigenvalue decomposition \( \Sigma_{rd} = U_{\Sigma_{rd}} \Lambda_{\Sigma_{rd}} U_{\Sigma_{rd}}^\dagger \). Then, optimum relay transmit covariance matrix \( Q_r \) has the spectral decomposition \( Q_r = U_{\Sigma_{rd}} \Lambda_{\Sigma_{rd}} U_{\Sigma_{rd}}^\dagger \) for any \( Q_r \) when there is only transmit covariance information at the transmitters and perfect CSI at the receivers.

**Proof:** Since \( Q_r \) only appears in \( I_{mac} \) but not in \( I_{sr} \), it is sufficient to consider \( I_{mac} \) only for the purposes of this proof. Let us define \( T = I + H_{sd} Q_s H_{sd}^\dagger \), we have

\[
\max_{Q_r} I_{mac} = \max_{Q_r} E \left[ \log |T + H_{rd} Q_r H_{rd}^\dagger| \right]
\]

(31)

Using the channel model, \( H_{rd} = Z_{rd} U_{\Sigma_{rd}} A_{\Sigma_{rd}}^{1/2} U_{\Sigma_{rd}}^\dagger \) can be inserted into (31). Noting that \( ZU \) and \( Z \) have the same joint distribution for zero mean identity covariance Gaussian \( Z \) and unitary \( U \), we have

\[
\max_{Q_r} E \left[ \log |T + Z_{rd} A_{\Sigma_{rd}}^{1/2} U_{\Sigma_{rd}}^\dagger Q_r U_{\Sigma_{rd}} A_{\Sigma_{rd}}^{1/2} Z_{rd}^\dagger| \right]
\]

(32)

The matrix between \( Z_{rd} \) and \( Z_{rd}^\dagger \) can be decomposed as \( A_{\Sigma_{rd}}^{1/2} U_{\Sigma_{rd}}^\dagger Q_r U_{\Sigma_{rd}} A_{\Sigma_{rd}}^{1/2} = UAU^\dagger \). Inserting this into (32) and using the fact that \( ZU \) and \( Z \) have the same joint distribution one more time, we have

\[
\max_{Q_r} E \left[ \log |T + Z_{rd} A Z_{rd}^\dagger| \right]
\]

(33)

Since the optimization problem in (33) does not involve \( U \), and choosing \( U = I \) does not violate the power constraint [4], we have \( Q_r = U_{\Sigma_{rd}} \Lambda_{\Sigma_{rd}} U_{\Sigma_{rd}}^\dagger \).

The result of Theorem 2 agrees with previous results in MIMO-MAC channels that shows that the eigenvectors of the transmit covariance matrix of each user are equal to the eigenvectors of its own channel covariance matrix. In MIMO relay channel as well, both the source and relay transmit along the eigenvectors of their own channel.

Having found the eigenvectors (i.e., transmit directions) of the source and relay transmit covariance matrices, next we find the optimum power values allocated along these transmit directions. The amount of power allocated in each direction depends on both transmit directions and the power allocations on the system. Next, we find the eigenvalues of the transmit covariance matrices. Re-writing (29) with transmit directions,
we have

\[
V(\alpha^*) = \max_{\lambda_1^{Q_s}, \lambda_1^{Q_r}} (\alpha^* I_{mac}(\lambda) + (1 - \alpha^*) I_{sr}(\lambda))
\]  

(34)

\[
I_{mac}(\lambda) = E \left[ \log \left( I + \sum_{i=1}^{M_s} \lambda_i^{Q_s} \phi_i^H \phi_i + \sum_{i=1}^{M_r} \lambda_i^{Q_r} \Sigma_{sd} \phi_i^H \phi_i \right) \right]
\]  

(35)

\[
I_{sr}(\lambda) = E \left[ \log \left( I + \sum_{i=1}^{M_r} \lambda_i^{Q_r} \Sigma_{sr} \phi_i^H \phi_i \right) \right]
\]  

(36)

where \( \lambda = [\lambda^{Q_s}, \lambda^{Q_r}] \) and \( \lambda^{Q_s} \) and \( \lambda^{Q_r} \) are the eigenvalue vectors of the source and relay transmit covariance matrices respectively, and \( \phi_i \) is the \( i \)-th column of the matrix \( \Phi = Z^{sd} \Lambda_s^{1/2} U_s^{sd} U_{sr}^* \).

The Lagrangian of (34) is given as

\[
R(\alpha^*, \lambda) = \mu_s \left( \sum_{i=1}^{M_s} \lambda_i^{Q_s} E_s^r(\lambda) - P_s \right) - \mu_r \left( \sum_{i=1}^{M_r} \lambda_i^{Q_r} E_r(\lambda) - P_r \right)
\]  

(37)

By taking the derivative of (37) with respect to \( \lambda_i^{Q_s} \) and \( \lambda_i^{Q_r} \), we obtain the Karush-Kuhn-Tucker (KKT) conditions for all \( 0 \leq i \leq M_s \) and \( 0 \leq j \leq M_r \):

\[
\alpha^* E \phi_i^H A^{-1} \phi_i + (1 - \alpha^*) \lambda_i^{Q_s} E_s^r B^{-1} z_i^{sr} \leq \mu_s \]  

(38)

\[
\alpha^* \lambda_i^{Q_r} E_s^{rd} z_i^{rd} A^{-1} z_i^{rd} \leq \mu_r \]  

(39)

where \( \mu_s \) and \( \mu_r \) are the Lagrangian multipliers, \( A \) and \( B \) are the matrices inside the determinants of (35) and (36) respectively. Let us denote the left hand side of (38) as \( E_i^s(\lambda) \) and the left hand side of (39) as \( E_i^r(\lambda) \). By multiplying both sides of (38) with \( \lambda_i^{Q_s} \) and (39) with \( \lambda_i^{Q_r} \) and summing over \( i \) and \( j \) respectively, we have

\[
\mu_s = \sum_{i=1}^{M_s} \lambda_i^{Q_s} E_i^s(\lambda), \quad \mu_r = \sum_{j=1}^{M_r} \lambda_j^{Q_r} E_j^r(\lambda)
\]  

(40)

Combining these with (38)-(39), we have the following fixed point equations for the eigenvalues of the source and relay transmit covariance matrices

\[
\lambda_i^{Q_s} = \frac{\lambda_i^{Q_s} E_i^s(\lambda) P_s}{\sum_{i=1}^{M_s} \lambda_i^{Q_s} E_i^s(\lambda)} \quad \lambda_j^{Q_r} = \frac{\lambda_j^{Q_r} E_j^r(\lambda) P_r}{\sum_{j=1}^{M_r} \lambda_j^{Q_r} E_j^r(\lambda)}
\]  

(41)

We propose the following iterative algorithm to solve for the above fixed point equations.

\[
\lambda_i^{Q_s}(n + 1) = \frac{\lambda_i^{Q_s(n)}(n) E_i^s(\lambda(n))}{\sum_{i=1}^{M_s} \lambda_i^{Q_s(n)}(n) E_i^s(\lambda(n))} P_s
\]  

(42)

\[
\lambda_j^{Q_r}(n + 1) = \frac{\lambda_j^{Q_r(n)}(n) E_j^r(\lambda(n))}{\sum_{j=1}^{M_r} \lambda_j^{Q_r(n)}(n) E_j^r(\lambda(n))} P_r
\]  

(43)

This iterative algorithm finds the optimum eigenvalues of the transmit covariance matrices of the source and relay nodes for Case 3. Finally, a minimization over \( \alpha \) is performed in order to find which case results in the lower bound.

It is important to note that the data rate resulting from classical convex optimization methods is equal to the lower bound to the capacity. However, the data rate resulting from the proposed algorithm might be less than the data rate resulting from classical convex optimization methods, and therefore the lower bound. In the Numerical Analysis section, we will simulate the data rate gap between two methods.

IV. NUMERICAL RESULTS

In this section, we simulate our proposed solution. The convergence analysis of the algorithms in Cases 1 and 2 are very similar to the convergence analysis in [5]. In Figure 2, Case 3 is considered and optimum eigenvalues of the transmit covariance matrix of the source and relay are plotted. Here, \( \lambda_1^{Q_s} \) is the first eigenvalue and \( \lambda_2^{Q_r} \) is the second eigenvalue of the source, \( \lambda_1^{Q_r} \) is the first eigenvalue and \( \lambda_2^{Q_r} \) is the second eigenvalue of the relay. The power constraints \( (P_s, P_r) \) are 10 dB. In figure, we observe that our algorithm converges to the optimum eigenvalues.

In this paper, the lower bound to the relay channel capacity is determined for three cases. In Figure 3, \( V(\alpha) \) curves with respect to \( \alpha \) are shown for Case 1 and Case 2. The power constraints \( (P_s, P_r) \) are 10 dB. We use CVX package for MATLAB [13] for implementing classical convex optimization methods. As expected, the results of the proposed algorithm are the same as classical convex optimization methods for Case
1 and Case 2. However, the simulation run time is much faster for the proposed algorithm.

For Case 3, $V(\alpha)$ curves with respect to $\alpha$ are shown in Figure 4. The minimum value of $V(\alpha)$ is the resulting data rate of the optimization problem. For the classical convex optimization method, the minimum of $V(\alpha)$ gives the lower bound to the capacity. For the proposed algorithm, the minimum of $V(\alpha)$ is almost the same as the lower bound. Therefore, we can say that the assumption that the link from the source to destination is weaker than the link from the source to relay is validated. At the optimum $\alpha^*$ value, the proposed algorithm does not lose much data rate with respect to the lower bound. On the other hand, the proposed algorithm gains significantly in terms of speed, as the classical convex optimization methods are extremely slow.

When the source power is fixed at 10 dB, the effect of increasing relay power is analyzed in Figure 5. We observe that the channel is subject to Case 2 condition when the relay power is 5-10 dB. When the relay power is 10-15 dB, the channel is subject to Case 3 condition. Lastly, the channel is subject to Case 1 condition when the relay power is 15-30 dB. The channel saturates with relay power since in Case 1 the relay power is large enough to forward all the information decoded at the relay node to the destination node, and the achievable rate is limited by the capacity of the source to relay link [9].

V. Conclusion

We analyzed a MIMO relay channel when the transmitters have partial channel state information and the receivers have perfect channel state information. The MIMO relay channel capacity for such a system is not known in general. In this paper, we defined a lower bound on the channel capacity which was in terms of a max-min optimization problem. To solve this problem, we combine our system model with the results in [9]. We found the transmit directions of the source and relay nodes, and in order to achieve best lower bound, we found the optimum power allocation policies (over the antennas) of the source and relay nodes. The solution of the max-min problem is given in three cases. For two of these cases, it is possible to propose efficient and fast algorithms that give the optimum source and relay transmit transmit covariance matrices. For the third case, most general situation can only be solved with classical convex optimization methods. However, by making a reasonable assumption on the relay channel, we propose an efficient and fast algorithm for the third case as well. This assumption is validated through simulations by showing that the data rate obtained by our proposed algorithm is almost the same as the data rate obtained by the classical convex optimization methods.

References