Interfaces with Other Disciplines

Solving knapsack problems with \textit{S}-curve return functions

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Abstract

We consider the allocation of a limited budget to a set of activities or investments in order to maximize return from investment. In a number of practical contexts (e.g., advertising), the return from investment in an activity is effectively modeled using an \textit{S}-curve, where increasing returns to scale exist at small investment levels, and decreasing returns to scale occur at high investment levels. We demonstrate that the resulting knapsack problem with \textit{S}-curve return functions is NP-hard, provide a pseudo-polynomial time algorithm for the integer variable version of the problem, and develop efficient solution methods for special cases of the problem. We also discuss a fully-polynomial-time approximation algorithm for the integer variable version of the problem.

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1. Introduction and motivation

The allocation of budget to competing activities occurs in nearly all business applications. When the budget limit is the only constraining factor (when, for example, funds may be invested in various instruments), then the resulting problem falls in the well-known class of knapsack problems. The standard 0–1 knapsack problem considers a number of items, each with a known weight and value, with a goal of maximizing the value obtained by selecting a subset of the items whose collective weight does not exceed a given capacity limit (see, e.g. [18]). In certain contexts (e.g., investment in various financial instruments) the effective weight of an item may itself be a decision variable. That is, if we are free to invest any non-negative amount up to some upper limit in each element of a set of investment instruments, then we have a continuous version of the knapsack problem. When the value of the instrument is linear in the amount invested, then the resulting problem is a continuous knapsack problem that can be solved by inspection: simply sort instruments in non-increasing order of per-unit revenue, and insert items into the knapsack until the capacity is exhausted (the more general standard continuous knapsack problem may employ a capacity consumption factor per-unit of the decision variable value, in which case we simply sort items in non-increasing order of the ratio of per-unit value to per-unit capacity consumption).

If the value of the investment instrument is not a linear function of the investment level, then the resulting non-linear knapsack problem is not necessarily easily solved (see [6] for a comprehensive review of the literature on non-linear knapsack problems). In a number of practical applications, including portfolio selection and advertising budget allocation, the return on investment function may take a non-linear form leading to complex classes of non-linear knapsack problems. The relationship between advertising budget allocation and sales response has served as the topic of many studies in marketing. Simon and Arndt [25] surveyed the characteristics of sales-advertising response functions. Their survey of the literature showed that the majority of research on advertising response subscribes to one of two proposed shapes of the response function: (1) a non-negative concave-downward curve and (2)
an \( S \)-curve. Thus, if a supplier’s sales response to advertising in each member of a set of markets follows one of these forms, the supplier faces the challenge of determining the amount of a limited budget to allot to each market in order to maximize sales. In sales-force time-management contexts, a similar phenomenon occurs, where the frequency of sales calls to a client affects the sales response. Lodish [16] characterizes this response as following an \( S \)-curve shape as a function of sales visit frequency. The salesperson must therefore allocate the number of available visits during a planning horizon to each member of a set of clients in order to maximize sales revenue.

Burke et al. [7] analyzed a related problem that considers the case of concave-downward response functions (which are characterized by non-negative concave functions with zero return at the origin). They focus on a setting in which a buyer must purchase a fixed quantity from a number of capacitated suppliers, and where each supplier offers a (concave) quantity discount structure. In contrast, we focus on the commonly employed \( S \)-curve return functions where increasing returns to scale exist at small investment levels, and decreasing returns to scale occur at high investment levels. Fig. 1 illustrates an example of the shape of the \( S \)-curve return functions we consider.

We examine a budget allocation problem requiring the best allocation of an available budget \( A \) among \( N \) independent instruments. The return of instrument \( i \) is given by the function \( \mu_i(a_i) \), where \( a_i \) is the investment level allocated to instrument \( i \). The objective is to maximize total net return from budget allocation to the different instruments while not exceeding the limited budget (we later define the term net return more precisely in Section 3). We recognize the potential uncertainties existing in such application areas, and our model can be employed in such contexts when each function \( \mu_i(a_i) \) represents the expected return for a given investment level, and when the objective is to maximize net expected return. We first show that the problem of allocating a budget among competing activities while maximizing the net return is NP-hard when the return function takes the form of an \( S \)-curve. We then consider global optimization methods for solving the problem, analyze a special case in which all \( S \)-curves have the same shape, and show that this special case can be solved in polynomial time. Following this, we examine the practical special case in which the investment levels must come from a discrete set of values, and provide a pseudo-polynomial time algorithm for this case, as well as a fully polynomial time approximation scheme (FPTAS).

The remainder of this paper is organized as follows. Section 2 next reviews related literature on budget allocation problems and applications. We define the problem and model formulation in Section 3, discuss solution approaches for this continuous optimization problem, and then present a polynomially solvable special case in Section 4. In Section 5, we consider a discrete variable version of the problem, providing a pseudo-polynomial time algorithm as well as a fully polynomial time approximation scheme. Finally, concluding remarks are provided in Section 6.

2. Literature review

The allocation of resources among different activities is a critical issue in almost all sectors, which has led to a multitude of research papers on this topic. We discuss related papers on work that deals with sales/advertising contexts, and also consider related work on non-linear knapsack problems. Zoltners and Sinha [30] provide a literature review and a conceptual framework for sales resource allocation modeling. They develop a general model for sales resource allocation which simultaneously accounts for multiple sales resources, multiple time periods and carryover effects, non-separability, and risk. Moreover, they discuss several actual applications of the model in practice, which illustrates the practical value of their integer programming models.

When the sales response or costs are not known with certainty, they are often characterized using probability distributions. Holthausen and Assmus [12] discuss a model for the allocation of an advertising budget to geographic market segments, when the sales response to advertising in each segment is characterized by a probability distribution. Their model derives an efficient frontier in terms of the expected profit and the variance resulting from alternative budget allocations. Norkin et al. [22] propose a general stochastic search procedure for the optimal allocation of indivisible resources, which is posed as a stochastic optimization problem involving discrete decision variables. The search procedure develops a branch-and-bound method for this stochastic optimization problem.

The problem of resource allocation among different activities, such as allocating a marketing budget among sales territories is analyzed by Luss and Gupta [17]. They assume that the return function for each territory uses different parameters, and derive single-pass algorithms for different concave payoff functions (based on the Karush–Kuhn–Tucker, or KKT, conditions) in order to maximize total returns for a given amount of effort. A number of efficient procedures have been developed subsequent to this for solving single-resource-allocation problems under objective function and constraint assumptions that lead to convex programming problems, including Zipkin [29].
Bitran and Hax [3], Brethauer and Shetty [4,5], and Kodialam and Luss [15]. In addition, several papers have focused on non-linear knapsack problems satisfying these convexity assumptions, when the variables must take integer values, including Hochbaum [11], Mathur et al. [19], and Brethauer and Shetty [4,5].

Surprisingly little literature exists on continuous knapsack problems involving the minimization of a concave objective function (where the KKT conditions are not sufficient for optimality). Moré and Vavasis [20] provide an efficient method for finding locally optimal solutions for this class of problems assuming objective function separability. Burke et al. [7] consider a problem in which a producer must procure a quantity of raw materials from a set of capacitated suppliers. The producer seeks to obtain its required materials at minimum cost, where each supplier provides a concave quantity discount cost structure. The resulting problem takes the form of a continuous knapsack problem involving the minimization of the sum of separable concave functions. They provide a pseudo-polynomial time algorithm and a fully-polynomial-time approximation scheme for the general version of the problem. Sun et al. [26] provide a partitioning method for the integer version of this problem that uses a linear underestimation of the objective function to provide lower bounds at each iteration. Romeijn et al. [23] consider the minimization of a specially structured non-separable concave function over a knapsack constraint, and provide an efficient algorithm for solving this problem.

The literature on knapsack problems in which the objective function is non-convex (and non-concave) is somewhat limited. Ginsberg [9] was the first to consider a knapsack problem with S-curve return functions, which he referred to as “nicely convex–concave production functions”. He characterized structural properties of optimal solutions assuming differentiability of the return functions, and predominantly assuming identical return functions. Lodish [16] considered a non-linear non-convex knapsack problem in a salesforce planning context in which the response function is defined at discrete levels of salesforce time investment. He approximated this problem using the upper piecewise linear concave envelope of each function, and provided a greedy algorithm for solving this problem (this greedy algorithm provides an optimal solution for certain discrete knapsack sizes, but not for an arbitrary knapsack size). Freeland and Weinberg [8] addressed the continuous version of this problem and proposed solving the approximation obtained by using the upper concave envelope of each continuous return function. Zoltners et al. [31] consider general response functions and also propose an upper concave envelope approximation, along with a branch-and-bound procedure, that permits successively providing better approximations of the continuous functions at each branch. We discuss a similar method for solving the continuous version of the problem with S-curve return functions that takes advantage of the specialized structure of these return functions. Morin and Marsten [21] devised a dynamic programming approach for discrete, non-linear, and separable knapsack problems, where no convexity or concavity assumptions are made on the objective function, which they assume to be non-decreasing in the decision variables. The storage requirements for determining dominated solutions in their approach grow exponentially, however, in the number of items. Romeijn and Sargut [24] recently considered a non-convex, continuous, and separable knapsack problem, which results as a pricing subproblem in a column generation approach for a stochastic transportation problem. They use a sequence of upper bounding functions that permits solving a sequence of specially-structured convex programs such that, in general, the procedure converges to an optimal solution in the limit (we discuss a similar approach for solving the continuous version of our problem in the next section).

Knapsack problems with non-convex (and non-concave) objective functions, such as those mentioned in the previous paragraph, fall into the difficult class of global optimization problems (see [13]), which require specialized search algorithms that often cannot guarantee finite convergence to a globally optimal solution. The S-curve functions we consider fall into this category, although we are able to exploit the special structure of these functions to provide effective methods for solving the discrete version of this problem. As we later discuss in greater detail, the continuous version of the problem we consider falls into the class of monotonic optimization problems [27], and specialized methods developed for this class of global optimization problems thus provide a viable option for providing good solutions.

Our primary contributions relative to this body of previous research include showing that the continuous knapsack problem with non-identical S-curve return functions is NP-hard, providing potential global optimization approaches for solving this difficult problem, and in providing both a pseudo-polynomial time algorithm and a fully polynomial time approximation scheme for the discrete version of the problem.

3. Problem description, formulation, and solution properties

We consider a set \( I = \{1, \ldots, N\} \) of marketing instruments, indexed by \( i \), such that the expected return from investing \( a_i \) dollars in instrument \( i \) is given by the function \( \mu_i(a_i) \) for each \( i \in I \). We assume that the function \( \mu_i(a_i) \) is non-negative and everywhere locally Lipschitz continuous for all \( a_i \geq 0 \), and that \( \mu_i(a_i) \) is a convex non-decreasing function for \( 0 \leq a_i \leq \beta \) and is concave for \( a_i \geq \beta \) for some non-negative \( \beta \). These functional properties allow for modeling S-curve return functions that arise in a number of applications. We define the net return function from investing \( a_i \), in instrument \( i \) as \( \mu_i(a_i) = \tilde{\mu}_i(a_i) - a_i \), and we assume \( \mu_i(0) \geq 0 \) for all \( i \in I \). Note that the function \( \mu_i(a_i) \) inherits the convexity and concavity properties of \( \tilde{\mu}_i(a_i) \). Let \( \partial \mu_i(a_i) \) denote the set of subgradients of \( \mu_i(a_i) \) at \( a_i \). Because \( \mu_i(a_i) \) is neither everywhere convex nor every-
where concave, we define these subgradients as follows. For \( a_i \in [0, \beta_i] \) (in the convex region of \( \mu_i(a_i) \)), if \( \xi \in \partial \mu_i(a_i) \), we have \( \mu_i(a_i) \geq \mu_i(\tilde{a}_i) + \xi(a_i - \tilde{a}_i) \) for any \( a_i \in [0, \beta_i] \). For \( a_i \in [\beta_i, \infty) \) (in the concave region of \( \mu_i(a_i) \)), if \( \xi \in \partial \mu_i(a_i) \), we have \( \mu_i(a_i) \leq \mu_i(\tilde{a}_i) + \xi(a_i - \tilde{a}_i) \) for any \( a_i \in [\beta_i, \infty) \).

We wish to allocate a budget of \( A \) dollars to the marketing instruments in order to maximize total expected return. We formulate this knapsack problem with S-curve return functions (KPS) as follows:

**[KPS]**

Maximize

\[
\sum_{i=1}^{N} \mu_i(a_i)
\]

Subject to:

\[
\sum_{i=1}^{N} a_i \leq A,
\]

\[
a_i \geq 0, \quad i = 1, \ldots, N.
\]

Note that we can apply a non-negative weight \( c_i \) to any item \( i \) in the objective function (e.g., \( \mu_i(c_i a_i) \)) by simply redefining our \( \mu_i \) function (i.e., \( \mu_i(a_i) = \mu_i(c_i a_i) \)), and the resulting functions retain the S-curve shape (we then need to redefine our \( \beta_i \) and \( \gamma_i \) values accordingly). We can also accommodate a non-negative weight \( c_i \) in the constraint (e.g., \( \sum_{i=1}^{N} c_i a_i \leq A \)) using the variable substitution \( a_i' = c_i a_i \) and redefining the net return function using \( \mu_i(a_i') = \mu_i(c_i a_i) \).

Next, we show that we can assume that the S-curves we will deal with are all non-decreasing and non-negative, without loss of generality. Let \( \gamma_i \) equal the minimum between \( A \) and the minimum value of \( a_i \) in the concave portion of \( \mu_i(a_i) \) at which point this function reaches its maximum value. We can assume that if \( \gamma_i = A \), then \( \mu_i(a_i) = \mu_i(\gamma_i) \) for all \( a_i \geq A \), and we therefore have that \( 0 \in \partial \mu_i(\gamma_i) \) for all \( i \in I \). Note that an optimal solution exists such that \( a_i \leq \gamma_i \) for all \( i \in I \), and we assume that \( \mu_i(\gamma_i) \geq 0 \). We can then assume without loss of generality that each \( \mu_i(a_i) \) is non-decreasing on \([0, \gamma_i]\) (any decreasing portion of the function must occur in the convex segment of the function, and we can redefine \( \mu_i(a_i) \), if necessary, using \( \mu_i(a_i) = \max\{\mu_i(0), \mu_i(a_i)\} \); the resulting functions continue to obey our S-curve properties). Since we assume that \( \mu_i(0) \geq 0 \) we therefore need only consider non-negative, non-decreasing functional forms for \( \mu_i(a_i) \) for all \( i \in I \). We next provide a key result on the complexity of [KPS].

**Theorem 1.** Problem [KPS] is NP-hard.

**Proof.** Consider an instance of the NP-hard 0–1 knapsack problem:

**[KP0,1]**

Maximize

\[
\sum_{i=1}^{n} r_i x_i
\]

Subject to:

\[
\sum_{i=1}^{n} w_i x_i \leq A,
\]

\[
x_i \in \{0, 1\}, \quad i = 1, \ldots, N,
\]

where \( A, r_i, w_i > 0 \) and \( A, w_i \) integer for all \( i \). Given an instance of [KP0,1], we can construct an instance of [KPS] as follows. Given item \( i \) define

\[
\mu_i(a_i) = \begin{cases} 
0, & 0 \leq a_i \leq w_i - 1, \\
\frac{r_i(a_i - (w_i - 1))}{w_i - 1}, & w_i - 1 \leq a_i \leq w_i, \\
r_i, & w_i \leq a_i.
\end{cases}
\]

Note that this definition of each \( \mu_i(a_i) \) is consistent with our assumptions on the \( \mu_i(a_i) \) functions in the definition of the problem [KPS] (with \( \beta_i \) taking any value on the interval \([w_i - 1, w_i]\)). Observe that for any \( a_i \leq w_i - 1 \), item \( i \)'s contribution to the objective function of [KPS] is zero, while for any \( a_i \geq w_i \), item \( i \) contributes \( r_i \) to the objective function. An optimal solution for this special case of [KPS] exists with all \( a_i \) integer (because \( A \) is integer and each \( \mu_i(a_i) \) is piecewise linear with integer breakpoints), and any optimal solution can thus be modified to a solution with equivalent objective function value where each \( a_i \) equals 0 or \( w_i \). Therefore, an optimal solution to this special case of [KPS] provides an optimal solution for the corresponding instance of [KP0,1], which implies the NP-hardness of problem [KPS].

**Solving [KPS].** The non-convexity of the S-curve functions and the above result imply that we need to draw on global optimization methods for solving [KPS]. We will discuss two such approaches: the first employs recent results on monotonic optimization problems, while the second exploits the special structure of the S-curve functions we are considering.

As mentioned in the previous section, problem [KPS] falls into the class of monotonic global optimization problems [27], because we are maximizing a non-decreasing function subject to a non-decreasing constraint limited by an upper bound (and where the variables are non-negative). Tuy [27] demonstrates the intuitive result that, for such problems, an optimal solution exists on the boundary of the feasible region. He proposes a so-called polyblock algorithm, which performs a search over a sequence of hyper-rectangles. We next briefly describe the application of this approach for solving [KPS]. Let \( a \) denote the vector of \( a_i \) values (\( i = 1, \ldots, N \)), and let \( a^L \) and \( a^U \) denote lower and upper bound vectors on \( a \) (initially we have \( a^L_i = 0 \) and \( a^U_i \) is the vector of \( \gamma_i \) values, where the subscript 0 corresponds to an iteration counter). Define \( \mathcal{A} \) as the set of all \( a \in \mathbb{R}^N \) that satisfy the budget constraint (1). Beginning with the initial interval (or poly-block) \( P_0 = [a^L_0, a^U_0] \), it is clear that (a) if \( a^U_0 \in \mathcal{A} \), then this solution is optimal (because of the monotonicity and boundary solution properties), and (b) if \( a^U_0 \notin \mathcal{A} \), then the problem is infeasible. Assuming that neither of these holds, we wish then to bisect this polyblock into two smaller poly-blocks along one of the variable dimensions. For example, if \( j \) denotes the index of the item with the maximum value of \( \gamma_i \), suppose we consider the two polyblocks \( P_{1,1} = [a^L_1, a^L_{1,1}] \) and \( P_{1,2} = [a^L_1, a^L_{1,2}] \), where \( a^L_{1,1} = a^L_0 \) for
all $i, a_{i}^{U} = a_{i}^{L}$ except for the $j$th element, which equals $\gamma_j/2$. Similarly, $a_{i}^{U} = a_{i}^{L}$ except for the $j$th element, which equals $\gamma_j/2$, while $a_{i}^{L} = a_{i}^{U}$. We now have two polyblocks whose union equals the initial polyblock $P_0$ and (whose intersection is empty except at the boundary).

Given any polyblock $P_k = [a_k^{U}, a_k^{L}]$, then clearly if $a_k^{L} \notin \mathcal{A}$, we can eliminate (prune) the polyblock; on the other hand, if $a_k^{L} \in \mathcal{A}$ then this solution provides both an upper and lower bound for the best possible solution in the polyblock. If neither of these holds, then $a_k^{L}$ serves as a lower bound on the best solution in the polyblock, and we utilize an upper bounding method for the best solution in the polyblock (this can be obtained, for example, by establishing the upper concave envelope of each of the functions $\mu_i(a_i)$ in [KPS], replacing these functions with this upper concave envelope in [KPS], and solving the resulting convex program; to do this, we simply determine the smallest point on the concave portion of $\mu_i(a_i)$ such that $\mu_i(a_i)/a_i \in \partial \mu_i(a_i)$, and connect a line from the origin to this point). We therefore have all of the elements we need for a branch-and-bound type of algorithm, where branching corresponds to bisecting a variable (and thus splitting a polyblock in two), and fathoming a polyblock with index $k$ is done by either (a) verifying that $a_k^{L}$ is feasible and therefore the best possible solution for the polyblock; (b) verifying that $a_k^{L}$ is infeasible, and thus pruning the polyblock, or (c) verifying that the polyblock’s upper bound solution value is inferior to the best known solution value. This polyblock algorithmic approach will either terminate with an $\epsilon$-optimal solution (where $\epsilon$ is a predetermined optimality tolerance), or will converge to an optimal solution value in the limit [27].

While the polyblock algorithm has been shown to be effective for monotonic optimization problems, the S-curve functions we consider have a special structure that we may exploit to provide alternative global optimization approaches for [KPS]. The following Theorems 2 and 3 provide important properties that we will utilize in developing an additional global optimization solution approach as well as solution methods for various special cases of [KPS]. In particular, Theorem 3 demonstrates that an optimal solution always exists such that at most one instrument $i$ will exist with positive investment at a level less than $\beta_i$ (i.e., in the convex portion of the $\mu_i(a_i)$ function). This theorem generalizes a similar result provided by Ginsberg [9] who considered the differentiable case with non-zero second derivatives (i.e., strict concavity in the concave portion and strict convexity in the convex portion of the function).

**Theorem 2.** In an optimal solution $a^*$ for [KPS], given any items $(i, j)$ such that $a^*_i, a^*_j > 0$, we must have $\partial \mu_i(a^*_i) \cap \partial \mu_j(a^*_j) \neq \emptyset$.

**Proof.** The proof follows from the necessity of the generalized Karush–Kuhn–Tucker (KKT) conditions (which are provided in the appendix); in particular, for all $i$ such that $a^*_i > 0$ the generalized KKT conditions require the existence of a non-negative $w$ such that $w \in \partial \mu_i(a^*_i)$.

**Theorem 3.** An optimal solution exists for [KPS] with $0 < a_i < \beta_i$ for at most one instrument $i$.

**Proof.** Consider an optimal solution $a^*$ with objective function value $z^*$ such that $0 < a^*_i < \beta_i$ and $0 < a^*_j < \beta_j$ for some $(i, j) \in I$. Note that by Theorem 2 we must have $\partial \mu_i(a^*_i) \cap \partial \mu_j(a^*_j) \neq \emptyset$. Consider a solution with $a_i = a^*_i$ for all $k \in I \setminus \{i, j\}$, $a_j = a^*_j + \delta$, and $a_i = a^*_i - \delta$ for some $\delta \leq \min\{\beta_j - a^*_j, \beta_j - a^*_i\}$, denote the objective function value of this new solution by $z_n$, and let $\xi^*$ denote an element of $\partial \mu_i(a^*_i) \cap \partial \mu_j(a^*_j)$. By the convexity of $\mu_i(a_i)$ for $0 \leq a_i \leq \beta_i$ (and of $\mu_j(a_j)$ for $0 \leq a_j \leq \beta_j$), we have

$$
\mu_i(a_i^* + \delta) \geq \mu_i(a_i^*) + \delta \xi^*,
$$

$$
\mu_j(a_j^* - \delta) \geq \mu_j(a_j^*) - \delta \xi^*.
$$

Considering the difference in objective function values between the two solutions given, we have

$$
z_n - z^* = \mu_i(a_i^* - \delta) - \mu_i(a_i^*) + \mu_j(a_j^* + \delta) - \mu_j(a_j^*)
$$

$$
\geq \delta (\xi^* - \xi^*) = 0.
$$

Since $z_n \geq z^*$, the new solution is optimal. Because this holds for any $\delta \leq \min\{\beta_i - a_i^*, \beta_j - a_j^*\}$, we can set $\delta = \min\{\beta_i - a_i^*, \beta_j - a_j^*\}$, which results in an optimal solution in which either $a_i = 0$ or $a_j = \beta_j$ (or both). The arbitrary selection of the indices $i$ and $j$ and the repeated application of this argument to any pair of items implies that the theorem holds.

**Theorem 3** allows us to eliminate the part of the feasible region where multiple items may take positive values strictly between 0 and $\beta_i$ in the convex portion of the net return function. This property becomes particularly useful in providing solution methods for a practical special case of problem [KPS] in Section 4. It can also aid in a more efficient application of global optimization techniques for [KPS]. We next discuss such a global optimization approach, which applies a method suggested by Zoltners et al. [31] for solving a linear relaxation of a non-linear sales resource allocation problem. This approach was also used by Romeijn and Sargut [24] for solving a linear relaxation of a singly-constrained non-linear pricing problem embedded in a stochastic transportation problem. This method considers successive upper concave approximations of the objective function of the original problem, and branches on a continuous variable for which the concave approximation is not coincident with the actual function value at the current solution. The successive upper concave approximations made after branching provide a closer approximation to the original function at each step (our subsequent discussion of the algorithm for [KPS] will demonstrate this property). Both Zoltners et al. [31] and Romeijn and Sargut [24] use this approach for solving integer programming problems, whereas the application we next describe involves continuous variables only. Because this method, in the worst-case, performs an exhaustive branch and bound search of the feasible region for optimal
continuous variable values, this method is able to find an ε-optimal solution in a finite number of steps, and is guaranteed to converge to a global optimal solution in the limit, although not finitely (see [31,24]).

This approach begins by considering the relaxation of [KPS] obtained by using the upper concave envelope of each S-curve function (see Fig. 2); note that the resulting relaxation is a convex program (and can therefore be efficiently solved using a commercial solver, for example). For this relaxation, Romeijn and Sargut [24] showed a similar result to Theorem 3, i.e., an optimal solution exists with at most one instrument taking a value on the part of the concave envelope that is not coincident with the actual function value (i.e., on the dashed line in the figure on the left in Fig. 2). This implies both that the relaxation will provide a reasonably close approximation to the original function value (and recall that an optimal solution exists such that this occurs for at most one item). Note that the resulting solution is feasible for [KPS], and we can evaluate this solution using the original objective function to provide a lower bound on the best solution for subproblem i. We then begin branching, with one branch (branch 1 in Fig. 2) considering solutions with \(0 \leq a_i \leq a_i^*\) and the other (branch 2 in Figure 2), solutions with \(a_i \geq a_i^*\). As in the solution at the root node, we use an upper linear approximation of the function within each of these intervals, as shown in the figure on the right in Fig. 2. Observe that each time we branch on item i, we provide a closer approximation of \(\mu(a_i)\), and that each problem considered at a node in the branch and bound tree is a convex program. Moreover, at each node, we obtain both upper and lower bounds on the optimal solution value. We can therefore use this branch and bound procedure in search of an ε-optimal solution. As with the polyblock algorithm, although we are only guaranteed to converge to a globally optimal solution in the limit, with either approach, we can generate a multitude of feasible solutions in reasonable computing time, with bounds on the deviation of each solution from optimality.

The following section discusses a special case in which the response functions obey certain strict relationships. These assumed relationships lead to a polynomial-time solution, and also allow us to explore the generalized KKT conditions for [KPS], which are necessary for local optimality of a solution (see, e.g., Hiriart-Urruty [10]).

4. Polynomially solvable special case

This section considers a special case in which the return functions obey certain properties that permit creating a preference rank ordering for different instruments. We first consider the case in which all of the return functions are identical, i.e., \(\mu(a_i) = \mu(a)\) with an inflection point occurring at \(\beta\) and a maximum value occurring at \(\gamma\) such that \(\beta < \gamma < A\) for all \(i \in I\). We restrict our focus to differentiable functions in this section (except possibly at \(\beta\)), noting that similar results can be obtained for functions with points of non-differentiability. Although this special case appears to be simple at first, its analysis allows us to illustrate the potential complexities of this problem class, and it also paves the way for characterizing the complexity of more general cases.

Under identical revenue curves, Theorem 2 implies that all instruments whose investment level is positive and falls in the concave part of the curve will have identical values of \(a_i\) at optimality. Moreover, Theorem 3 allows us to arbitrarily select any instrument as one whose \(a_i\) value may be positive and fall in the interval \((0, \beta)\). We employ the necessary KKT conditions (see Appendix) to analyze this problem. We first suppose that the KKT multiplier associated with the knapsack constraint, denoted by \(w\), is zero. In this case we have that \(\lambda_i = \partial \mu(a_i)/\partial a_i\) and \(\lambda_i a_i = 0\) for all \(i \in I\), where \(\lambda_i\) is a KKT multiplier associated with the \(i\)th non-negativity constraint. Thus if \(a_i\) is positive, we have that \(\partial \mu(a_i)/\partial a_i = 0\) at a KKT point when \(w = 0\). Because we assume (without loss of generality) that the return functions are non-decreasing, any zero derivative point in the convex portion of the curve must have a return function equal to \(\mu(0)\), and we can thus ignore stationary points in the convex portion of the curve. Noting that \(\partial \mu(\gamma)/\partial \gamma = 0\), and letting \(\bar{n} = [A/\gamma]\), we have that any solution such that \(\bar{n}\) of the \(a_i\) values are set to \(\gamma\) serves as
a candidate for an optimal solution (because each of these has objective function value $\bar{n}\mu(\gamma)$, we need only consider one such solution).

We next consider the case in which $w > 0$, which implies that the knapsack constraint must be tight at any associated KKT point. Such a KKT point must satisfy the following system of equations:

$$w = \frac{d\mu(a_i)}{da_i}, \quad \forall i \in I : a_i > 0,$$

$$\sum_{i=1}^{N} a_i = A, \quad w \geq 0, \quad a_i \geq 0, \quad \forall i = 1, \ldots, N.$$  

(Note that for our $S$-curves, $\frac{d\mu(a)}{da} \geq 0$ for all $a \geq 0$ and therefore $w \geq 0$ is redundant in the above system.) Recall that at most one variable can have an $a_i$ value that falls in the convex portion of the return function. Let us first consider cases in which no variable takes a value in this range. If we suppose that we know that $n$ variables take a positive value, then we seek a positive (single variable) $a$ such that $a \geq \beta$ and $na = A$. For $n = 1, \ldots, N$, we can quickly determine whether the solution with $a = A/n$ satisfies $d\mu(a)/da \geq 0$ with $a \geq \beta$. If such a solution exists, then it serves as a candidate for an optimal solution. We next suppose that one of the variables may take a value in the interval $(0, \beta)$. Letting $a'$ denote the value of the single variable in this interval, and letting $a''$ denote the value of the variable(s) falling in the concave portion, and noting that at most $N$ such solutions may exist (with $n = 0, \ldots, N - 1$), we seek solutions satisfying the following system of equations:

$$\frac{d\mu(a')}{{da'}} - \frac{d\mu(a'')}{{da''}} = 0,$$

$$a' + na'' = A,$$

$$0 \leq a' \leq \beta,$$

$$\beta \leq a''.$$

The difficulty of finding a solution to this system of equations depends on the functional form of the derivative function. In cases where the equation $d\mu(a')/da' - d\mu(a'')/da'' = 0$ intersects the line $a' + na'' = A$ only once in the interval $0 \leq a' \leq \beta$, for the given value of $n$ we can perform a line search to determine the unique solution satisfying the above system of equations. When the $\mu(a)$ function takes a second-degree polynomial form on both the convex and concave intervals, then this provides a sufficient condition for having at most one solution to the above system (note that if the equation $d\mu(a')/da' - d\mu(a'')/da'' = 0$ is linear, then it cannot be collinear with the equation $a' + na'' = A$ because $d\mu(a'')/da'' \geq 0$, eliminating the possibility of an infinite number of candidate solutions satisfying the KKT conditions). In such cases, the first constraint in the above system is linear, and we have two linearly independent equality constraints in two variables.

Assuming the previously stated conditions for a unique solution to the above system of equations, we then perform a line search for each possible value of $n$ along the line $0 \leq a' = A - na'' \leq \beta$ to determine (at most) $N$ additional candidate solutions. Note that if $\mu(a)$ is not strictly positive for all $a \in (0, \beta)$, then letting $\bar{a}$ denote the largest value of $a$ such that $\mu(a) = 0$, we can limit our search to the interval $(\bar{a}, \beta)$. The complexity of this line search is $\mathcal{O}(\log \beta)$. The overall complexity of this approach is therefore $\mathcal{O}(N \log \beta)$.

The following algorithm summarizes our approach for solving [KPS] with identical response functions, assuming the system of Eq. (3) has at most one solution for any value of $n$.

<table>
<thead>
<tr>
<th>Algorithm 1: Solve [KPS] with identical response functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initialize $LB = \bar{n}\mu(\gamma)$, where $\bar{n} = \lceil \frac{1}{\gamma} \rceil$</td>
</tr>
<tr>
<td>FOR $n = 1$ to $N$</td>
</tr>
<tr>
<td>.Set $a_n = \frac{A}{n}$</td>
</tr>
<tr>
<td>if $a_n \geq \beta$ and $n\mu(a_n) &gt; LB$ then</td>
</tr>
<tr>
<td>$LB \leftarrow n\mu(a_n)$</td>
</tr>
<tr>
<td>end if</td>
</tr>
<tr>
<td>Solve system of Eq. (3)</td>
</tr>
<tr>
<td>if a feasible solution exists for (3) with $a' = a'_n$, $a'' = a''_n$, and $\mu(a'_n) + n\mu(a''_n) &gt; LB$ then</td>
</tr>
<tr>
<td>$LB \leftarrow \mu(a'_n) + n\mu(a''_n)$</td>
</tr>
<tr>
<td>end if</td>
</tr>
<tr>
<td>end for</td>
</tr>
</tbody>
</table>

Optimal Solution Value $z^* = LB$

Note that this solution approach applies not only when all $N$ curves are strictly identical, but generalizes to the case in which each pair of curves differs by a constant value for all $a \geq 0$. In such cases, we cannot arbitrarily select an item whose value falls in the convex portion, but we need to separately consider each of the $N$ curves as a candidate for taking a value in the convex portion. Moreover, given that some $n$ items take values in the concave portion of the curve, we now have a dominance order in which to select these $n$ items from among the $N$ (i.e., those with higher values of $\mu(0)$ are preferred to those with lower values). For this case, the complexity therefore increases by a factor of $N$, leading to a worst-case bound of $\mathcal{O}(N^2 \log \beta)$.

For more general versions of the problem, where the $\mu_i(a)$ functions have no relationship, we showed in the previous section that the problem takes the form of a difficult global optimization problem. Although the KKT conditions are necessary for optimality, it is impractical in the general case to try to enumerate all KKT points in order to account for all local minima, in an attempt to find a global minimum. We next illustrate this complexity for the differentiable case. For the case in which the KKT multiplier for the knapsack constraint is zero (i.e., $w = 0$), we still require $d\mu(a)/da = 0$ for all $a > 0$. Because $d\mu(\gamma_i)/da = 0$, we require finding a subset $\mathcal{T}$ of $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ such that $\sum_{i \in \mathcal{T}} \gamma_i \leq A$ with the maximum value of $\sum_{i \in \mathcal{T}} \mu(\gamma_i)$. This problem is itself a 0–1 knapsack problem.
and, therefore, identifying candidate solutions using the KKT conditions does not lead to a polynomial-time solution approach. Additionally, the case in which \( w > 0 \) requires finding all solutions of a set of \( N \) (possibly non-linear) equality constraints (while also satisfying \( 2N \) non-negativity conditions), for each of the \( N \) choices of the variable which may take a value in the convex portion of the curve. We note here that non-linear programming methods used in a number of commercial solvers (such as conjugate gradient methods; see, e.g., Bazaraa et al. [2]) can be utilized in an attempt to identify a locally optimal point, although these methods cannot guarantee finding a globally optimal solution for global optimization problems.

We next focus on the practical case where all \( a_i \) variables must take integer values, where we can employ our previous properties of optimal solutions and provide algorithms of practical use that lead to solutions with provable bounds on performance.

5. KPS with integer variable restrictions

In the majority of practical contexts where problem [KPS] applies, it is acceptable (or even necessary) to limit the set of possible \( a_i \) values to a discrete set (where, e.g., each \( a_i \) value is denominated in some currency). Therefore, we henceforth consider problem [KPS] with the added restriction that each \( a_i \) value must be integer, denoting this problem as [KPSI] (note that our solution approaches can apply, it is acceptable (or even necessary) to limit values to a discrete set (where, e.g., each \( a_i \) value is denominated in some currency)). Therefore, we henceforth consider problem [KPS] with the added restriction that each \( a_i \) value must be integer, denoting this problem as [KPSI] (note that our solution approaches can easily be extended to any discrete candidate set of \( a_i \) values). We assume that the piecewise-linear function obtained by connecting successive values of \( \mu_i(a_i) \) at integer values of \( a_i \) with a line segment continues to satisfy the the \( S \)-curve properties we have defined. As the following theorem shows, we can work with these continuous piecewise-linear functions in order to solve [KPSI].

Let [KPSPL] denote the restricted version of [KPS] in which all of the \( \mu_i(a_i) \) functions are piecewise-linear functions with integer breakpoints.

**Theorem 4.** An optimal solution for [KPSPL] exists in which all \( a_i \) take integer values.

**Proof.** Assume we have an optimal solution \( a^* \) in which there are two or more fractional values of \( a_i \). Consider two of these fractional items \( a_i^j \) and \( a_i^k \) where \( l_j < a_i^j < u_j \) and \( l_k < a_i^k < u_k \), where \( (l_i, u_i) \) define the integer breakpoints on either side of \( a_i^* \) for any \( i \in I \). Let \( a_i = a_i^* \) for \( i \neq j, k \) and if we denote \( \overline{A} = A - \sum_{i \neq j, k} a_i \). Consider the following linear program (LP):

**Maximize** \( \mu_j(a_j) + \mu_k(a_k) = \rho_j a_j + \rho_k a_k \)

**Subject to:**

\[
\begin{align*}
    & l_j \leq a_j \leq u_j \\
    & l_k \leq a_k \leq u_k \\
\end{align*}
\]

where \( \rho_j \) and \( \rho_k \) are the slopes of the return functions of instrument \( j \) and \( k \) at the points \( a_i^j \) and \( a_i^k \), respectively. An optimal solution exists for this LP such that at least one of the variables \( j \) and \( k \) falls at one of its (integer) bounds (because at most one of these variables can be basic) and a resulting objective function value at least as high as that of the solution \( a^* \). Therefore, we either have that the original solution is suboptimal (a contradiction), or an alternative optimal solution is obtained with one less fractional variable. If two or more fractional variables remain, we can repeat this procedure until two fractional variables remain, whose sum must be integer because of the integrality of \( A \). An optimal solution for this final LP exists with integer values for both variables, as they must sum to an integer value, and at least one of them must take an integer value. \( \square \)

Theorems 3 and 4 together imply that we can solve the integer variable version of the problem using the continuous piecewise-linear function obtained by connecting successive values of \( \mu_i(a_i) \) at integer values of \( a_i \) with a line segment for all \( i \), and an optimal solution will exist with at most one value of \( a_i \) strictly between 0 and \( \beta_i \). This permits the construction of a pseudo-polynomial time algorithm for solving [KPSI] as we next discuss.

5.1. Pseudo-polynomial time algorithm

Given that an optimal solution exists with at most one instrument in the convex part of the \( S \)-curve, we can use the following approach. Suppose we arbitrarily select any instrument and assume that the investment amount for this instrument will be at some value in the convex portion of the function. That is, given some variable \( j \), suppose we set \( 0 < a_j < \beta_j \). This implies that any other variable \( i \) must either equal zero or fall in the interval \([\beta_i, \gamma_i]\). Given some value of \( a_j \) between 0 and \( \beta_j \), say \( d_j \), then define \( \overline{A}_j = A - a_j \). The remainder of the problem reduces to the following generalization of the 0–1 knapsack problem:

**[KPEI] Maximize** \( \sum_{i \in I \setminus \{j\}} \mu_i(a_i) \)

**Subject to:**

\[
\begin{align*}
    & \beta_i x_i \leq a_i \leq \gamma_i x_i, \quad \forall i \in I \setminus \{j\}, \\
    & a_i \in \mathbb{Z}_+, x_i \in \{0, 1\}, \quad \forall i \in I \setminus \{j\},
\end{align*}
\]

where \( \mathbb{Z}_+ \) is the set of non-negative integers. Balakrishnan and Geunes [1] provided a pseudo-polynomial time algorithm for the above problem when each \( \mu_i(a_i) \) has a fixed plus linear structure (i.e., a fixed reward for including item \( i \), plus a variable contribution to profit per unit weight). They referred to this problem as a knapsack problem with *expandable items*. Constraint (5) serves as a simple knapsack constraint. Constraint set (6) forces an item’s weight to zero if the item is not included in the knapsack (when \( x_i = 0 \)) and requires the item’s weight to fall between some prespecified upper and lower bounds if the item is included. The objective function (4) maximizes the net return from
filling the knapsack. The dynamic program used to solve
[KPEI] in [1] is a straightforward generalization of the standard
dynamic program used for solving knapsack problems,
where all integer feasible values of each \(a_i\) are
implicitly enumerated. The worst-case running time for this
dynamic programming approach for a given instrument \(j\)
assumed to have an investment level between 0 and \(b_j\)
and a given value of \(a_j\) is \(O(\text{NAT})\), where \(T = \max_{i \in I}\{\gamma_i - \beta_i\}\).

Using Theorems 3 and 4, we can solve [KPSI] by using
this dynamic programming approach to solve [KPEI] for
each possible value of \(a_i\) such that \(0 < a_i < \beta_i\) and for all
\(i \in I\). That is, we select an instrument \(j\), assume that
this instrument has an investment level between 0 and \(b_j\) (recall
that this can be true for at most one instrument), and solve
the associated problem [KPEI]. Because there are no more
than \(\bar{\beta} = \max\{\beta_i\}\) possible values for \(a_i\), and because we
must solve an instance of [KPEI] for each of these values of \(a_i\) and for each of the \(N\) items, the worst-case computa-
tional effort for this approach is \(\Theta(N^2A\bar{\beta}T)\).

5.2. Fully polynomial time approximation algorithm

We next develop a fully polynomial-time approximation
scheme (FPTAS) for [KPSI]. Given an \(\epsilon > 0\), an FPTAS
is polynomial in \(N\) and \(1/\epsilon\), and results in an objective
function value of no less than \((1 - \epsilon)z^*\), where \(z^*\) denotes the
value of the optimal solution. The approach we use is
related to the approach van Hoesel and Wagelmans [28]
employed for capacitated economic lot-sizing.

We begin with a “profit-based” dynamic program for
solving KPSI, letting \(F_i(\pi)\) denote the minimum amount
of capacity in the constraint consumed while providing a
profit of at least \(\pi\), and including all instruments up to
(and including) instrument \(i\). Thus, the the largest value
of \(\pi\) such that \(F_N(\pi) \leq A\) is the optimal solution value.
We assume a given upper bound on profit of \(\Pi\) (we will
subsequently describe our method for deriving a valid
value for this upper bound).

The dynamic program first solves the single-item
problem:
\[
F_1(\pi) = \min\{a_1|\mu_1(a_1) \geq \pi, 0 \leq a_1 \leq \gamma_1\}; \quad \pi = 0, \ldots, \Pi.
\]

Note that because \(\mu_1(a_1)\) is a univariate non-decreasing
function, the feasible region of the above problem is convex
(assuming it is non-empty). If \(\mu_1(0) > \pi\), then \(\mu_1(a_1) \geq \pi\)
for all \(a_1 \in [0, \gamma_1]\) and the constraint \(\mu_1(a_1) \geq \pi\) is redun-
dant (because \(\mu_1(a_1)\) is non-decreasing), and in this case
we have \(F_1(\pi) = 0\). If this is not the case, let \(a_1^*\) denote
the smallest value on the interval \([0, \gamma_1]\) such that
\(\mu_1(a_1^*) = \pi\), and thus \(F_1(\pi) = a_1^*\). We can, therefore, easily
determine \(F_1(\pi)\) for a well defined non-decreasing function
\(f_1\), by finding the (smallest) root of \(\mu_1(a_1) - \pi = 0\). This can
be done (assuming a root exists, which occurs if
\(\mu_1(\gamma_1) \geq \pi\)) in \(\Theta(\log \gamma_1)\) time using binary search. Next
consider the two-item problem:

\[
F_2(\pi) = \min_{a=0,1, \ldots, a_2=0,1, \ldots, \gamma_2} \{F_1(a) + a_2|\mu_2(a_2) \geq \pi - a\}.
\]

Given a value of \(\pi\), the inner minimization problem can be
solved in \(\Theta(\log \gamma_2)\) time. The general recursion can be sta-
ted as

\[
F_i(\pi) = \min_{a=0,1, \ldots, a_i=0,1, \ldots, \gamma_i} \{F_{i-1}(a) + a_i|\mu_i(a_i) \geq \pi - a\},
\]

with the inner minimization problem for a given \(a\) and \(\pi\)
having worst-case complexity \(\Theta(\log \gamma_i)\). Given \(i\) and \(\pi\), we can compute \(F_i(\pi)\) in \(\Theta(\pi \log \gamma_i)\) operations (where
\(\Gamma = \max\{\gamma_i\}\)). Because we have \(\Pi\) values of \(\pi\) and \(N\) values of \(i\), this recursion takes \(\Theta(N\Pi^2\log \Gamma)\) time. We thus have an algorithm that is pseudo-polynomial in the profit upper
bound value \(\Pi\).

We can set a suitable value of \(\Pi\) as follows. Letting
\(\mu^{\max} = \max\{\mu_i(\gamma_i)\}\), then \(\Pi = N\mu^{\max}\) provides an upper
bound on \(z^*\), the optimal solution value. Note also that
any single value of \(\gamma_i\) is feasible (i.e., less than or equal to
\(A\)), and we therefore have that \(\mu^{\max} \leq z^*\), which implies that
\(\Pi \leq Nz^*\).

Profit scaling. In the recursion we have described, sup-
pose that, instead of evaluating every integer value of \(\pi\), we
evaluate every \(K^{th}\) value. That is, for \(i \in I\) and
\(\pi \in [0, K, 2K, \ldots, [\Pi/K]\), define \(G_N(\pi)\) as the minimum
capacity that can be consumed using the first \(i\) instruments,
with a profit of no less than \(\pi\), when the profit allocated to
each instrument is a multiple of \(K\). Computing \(G_N(\pi)\) for
\(i \in I\) and \(\pi \in [0, K, 2K, \ldots, ([\Pi/K])\) then requires
\(\Theta(N\Pi^2\log \Gamma)\) operations. The following proposition
shows that using this approach provides a feasible solution,
and a bound on the objective function value.

Proposition 1. At least one element \(\pi \in \{0, K, 2K, \ldots, \lfloor \Pi/K \rfloor\} \exists\) with \(G_N(\pi) \leq A\), i.e., evaluating every \(K^{th}\)
value of \(\pi\) leads to a feasible solution if one exists. Moreover,
the maximum \(\pi\) in this set with \(G_N(\pi) \leq A\) equals at least
\(z^* - NK\).

Proof. Suppose we have an optimal solution with objective
value \(z^*\), and denote \(r_i\) as the profit contribution of
item \(i\) in this solution. Suppose we allocate a profit equal to
\(|r_i/K|K \leq r_i\) to item \(i\) (for all values of \(i\)). Then, in the
resulting solution, each value of \(a_i\) will be less than or equal
to its corresponding value in the optimal solution. The
resulting solution under the scaling algorithm is therefore
feasible under these profit allocations. Since \(\sum_{i \in I} r_i \leq \Pi\),
the following inequalities show that we account for these
profit allocations in the algorithm:

\[
\sum_{i \in I} \frac{r_i}{K}K \leq \sum_{i \in I} \frac{r_i}{K}K \leq \frac{\Pi}{K}K \leq \frac{\Pi}{K}K.
\]

Finally, by the definition of \(r_i\), we have

\[
\sum_{i \in I} |r_i/K|K = \sum_{i \in I} (|r_i/K| - 1)K \geq \sum_{i \in I} r_i/KK - NK = [z^*/K]K - NK.
\]
This last term is greater than or equal to \( z^* - NK \), which implies the stated result. \( \square \)

**Selecting an appropriate value of \( K \).** Suppose we select \( K = \max \{1, \frac{|\Pi|}{Nz^*} \} \). From Proposition 1, we have a lower bound on the solution value from the rounding procedure of \( z^* - NK \). Because \( NK = N \frac{|\Pi|}{Nz^*} \) and because \( \Pi \leq Nz^* \), we have \( NK \leq \varepsilon^* \), which implies that our lower bound satisfies \( z^* - NK \geq z^* - \varepsilon^* \), i.e., that our scaling approach finds a solution with value at least \( z^*(1 - \varepsilon) \). Recall our worst case complexity bound of \( O(\Pi(K/K)^s \log K) \). If \( \Pi/K < N \), then this results in a polynomial-time algorithm. Suppose, conversely, that \( \Pi/K > N \). If \( \frac{\Pi}{N} > 1 \), then \( K = \frac{|\Pi|}{Nz^*} \) if \( \frac{\Pi}{N} < 1 \), then \( K = 1 \). In either case we have \( K \geq \frac{|\Pi|}{Nz^*} \), which implies \( \Pi \leq \frac{2K^2}{z^*} \) and our worst-case complexity becomes \( O(\frac{2K^2}{z^*} \log K) \), which is polynomial in \( N \) and \( 1/\varepsilon \).

### 6. Concluding remarks

In this paper we discussed a single-resource allocation problem with non-linear \( S \)-curve returns as a function of resource allocation. The resulting model is a specially structured class of non-linear knapsack problems in which the objective function is neither concave nor convex. The structure of the \( S \)-curve, however, leads to a characterization of optimal solution properties that permits the development of practically useful solution algorithms. Because limited previous work exists on this class of relevant problems, we provided a pseudo-polynomial time solution algorithm, as well as a polynomial-time approach for a special case in which the return functions differ by a constant at all investment levels. We also provide a fully polynomial time approximation scheme, which, given a solution tolerance \( \varepsilon \), permits obtaining a solution within \( \varepsilon \% \) of an optimal solution value using an algorithm that is polynomial in the number of investment instruments. Future avenues for research include explicit consideration of the logistic functional forms found in much of the marketing literature on \( S \)-curves (see, e.g. [14]). A further exploration of the system of equations defined by the generalized KKT conditions might also provide value in the development of algorithms for the general form of the problem where investment levels may take any real-valued number.

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### Appendix. Generalized Karush–Kuhn–Tucker optimality conditions

When each \( \mu_i(a_i) \) function is locally Lipschitz continuous, the generalized KKT conditions for [KPS] are necessary but are not sufficient for optimality (see [10]). We will refer to a point that satisfies the generalized KKT conditions as a “KKT point”, and let \( \lambda_i \) be KKT multipliers associated with the lower-bound (non-negativity) constraints (2). Define \( \partial \mu_i(a_i) \) as the set of subgradients of the function \( \mu_i(\cdot) \) at \( a_i \), with \( \partial \mu_i(a_i) \) denoting the right directional derivative at \( a_i \) and \( \partial \mu_i(a_i) \) denoting the corresponding left directional derivative. For our \( S \)-curves, the set of subgradients at \( a_i \) is equal to the interval \([\partial \mu_i(a_i), \partial \mu_i(a_i)]\) if \( a_i \) lies in the convex portion of the function, while the set of subgradients at \( a_i \) equals the interval \([\partial \mu_i(a_i), \partial \mu_i(a_i)]\) if \( a_i \) lies in the concave portion of the function. The generalized KKT conditions can be written as:

\[
-\partial \mu_i(a_i) + w - \lambda_i \geq 0, \quad \forall i = 1, \ldots, N,
\]

\[
w \left( \sum_{i=1}^n a_i - A \right) = 0, \quad \lambda_i a_i = 0, \quad \forall i = 1, \ldots, N,
\]

\[
\sum_{i=1}^n a_i \leq A, \quad w \geq 0, \quad \lambda_i \geq 0, \quad \forall i = 1, \ldots, N.
\]

### References


